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Remarks on the Theory of Collective Choice

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Ever since the so-called paradox of voting was generalized by Arrow (1963) to every democratic method of collective decision-making, a vast literature has appeared (a) trying to circumvent Arrow's difficulty by weakening some of his conditions (Bordes, 1976; Hansson, 1973; Plott, 1973; Sen, 1969); (b) proposing some other paradoxes in the theory of collective choice (Batra and Pattanaik, 1972; Hansson, 1969; Schwartz, 1970; Sen, 1970a) and (c) casting doubts about the relevance of Arrow's theorem to the theory of Paretian welfare economics (Bergson, 1966; Little, 1952; Samuelson, 1967, 1977). The purpose of this paper is to make some remarks on these recent developments in the theory of collective choice.

The first part of the paper deals with the question of how much one needs to weaken Arrow's collective rationality condition in order to avoid his impossibility result. As is well known, Arrow (1963) imposed the collective rationality condition that the society can arrange all conceivable alternatives in order of preference and that, if some available set of alternatives is specified, the society must choose therefrom the best alternative with respect to that preference ordering. We will consider two conditions of consistent choice which are weaker than that of Arrow. The first condition requires that, if an alternative x is chosen over another alternative y in binary choice, y should never be chosen from any set of alternatives that contain x; while the second condition requires that, if x is chosen over y in binary choice, there exists no choice situation in which y is chosen and x is available but rejected. (In the second case y can be chosen if x is also chosen, while in the first case y cannot be chosen anyway.) There seems to be a gulf that separates possibility from impossibility in between these two seemingly similar consistency conditions. It will be shown that the first consistency requirement is *incompatible* with essentially Arrovian conditions on the collective choice rule, while the second consistency condition is *compatible* with the same conditions. Although the difference between these consistency requirements is very subtle, the implication thereof in the context of impossibility result is therefore dramatically different.

Lest we should be too satisfied, we must hasten to add that no collective choice rule satisfying our second consistency requirement can be free from the paradox of Paretian liberal (Sen, 1970a; Batra and Pattanaik, 1972).

The Arrow–Sen theory is then contrasted with the Bergson–Samuelson one. In view of doing this, it is convenient to remember that Arrow's incompatible conditions on collective choice rule can be classified into two categories. The first category consists of statements that apply to any *fixed* profile of individual preference relations, while the second category refers to the responsiveness of the collective choice to the *variations* in profiles. (The first category embraces the condition of *collective rationality* and the *Pareto rule*, while the second category consists of the *independence of irrelevant alternatives* and *non-dictatorship*. See Sections I and II below for the definition of these conditions.) Bergson (1966), Little (1952) and Samuelson (1967, 1977) agreed with Arrow so far as his conditions of the first category were concerned. It was only when Arrow went on to introduce some conditions of the second category that Bergson, Little and Samuelson come to deny the reasonableness and/or necessity thereof. Let us, therefore, fix a profile of individual preference relations. What we call the *Bergson–Samuelson social welfare ordering* is an ordering R such that, if a state x is Pareto-non-inferior (resp. Pareto-superior) to a state y with respect to the given profile, then x is not less preferred (resp. preferred) to y in terms of R. (Incidentally, the Bergson–Samuelson social welfare function is a numerical function u such that $u(x) \geqslant u(y)$ if and only if xRy for all states x and y.) Now we raise the problem of whether we can define such a social welfare ordering corresponding to a given profile. The answer is in the affirmative if each and every individual preference relation satisfies a strong consistency condition of transitivity. Our main purpose in this part is to strengthen this in that a social welfare ordering exists if and only if the Paretian unanimity rule corresponding to the given profile satisfies what we call the *axiom of consistency*. (Thus, even if intransitive individual preference is countenanced, we may still have a well-defined social welfare ordering.) This will be done by proving a general theorem on the extension of a binary relation.

What emerges as a result of our investigation is an ever sharper contrast between the *variable* profile framework of the Arrow–Sen theory, on the one hand, and the *fixed* profile framework of the Bergson–Little–Samuelson theory, on the other.

All the proofs are relegated into the final section which can be neglected by those who are not interested in technical details.

I. Rationality and Revealed Preference

For the sake of logical clarity, we discuss in this section the concept of rationality and that of revealed preference in abstraction from the problem of collective decision. The heart of our argument is the implication diagram given at the end of this section.

Let X be the set of all alternatives that are mutually exclusive. We assume that X contains at least three distinct elements. Also let K stand for the family of non-empty subsets of X, containing all the pairs and all the triples taken from X (and possibly more). A preference relation is a binary relation on X. Let R be a binary relation. By xRy (or, equivalently, $(x, y) \in R$) we mean that x is at least as good as y. By xPy and xIy we mean, respectively, (xRy and not yRx) and (xRy and yRx). An R is said to be *reflexive* if (xRx for all $x \in X$), *connected* if (xRy and/or yRx for all $x, y \in X$), *transitive* if (xRy and $yRz \rightarrow xRz$ for all $x, y, z \in X$), and *acyclic* if ($x^1Px^2 \ldots x^tPx^1$ for no finite subset $\{x^1, x^2, \ldots, x^t\}$ of X). A *quasi-ordering* is a reflexive and transitive relation and an *ordering* is a connected quasi-ordering.

Given a preference relation R and an $S \in K$, we define:

$$(1) \qquad G_R(S) = \{x : x \in S \text{ and } xRy \text{ for all } y \in S\}.$$

Clearly $G_R(S)$ is a subset of S such that $x^* \in G_R(S)$ has the following property: x^* is at least as good as any alternative in S.

A *choice function* on K is a function C which assigns a non-empty subset $C(S)$ of S to each $S \in K$. (It is intended that $S \in K$ represents a set of available alternatives and $C(S)$ represents the set of chosen elements from S.) We say that C is *rational* (R) if there exists a preference relation R, to be called a *rationalization* of C, such that

(2) $\quad C(S) = G_R(S)$ for all $S \in K$.

In other words, a choice function is rational if it can be construed as a result of preference optimization. (It should be noted that the concept of rational choice in itself has nothing to do with the transitivity of a rationalization.) We say, in particular, that C is *regular-rational* (RR) if (a) C is rational and (b) a rationalization thereof is an ordering. Arrow (1959) has shown that C is (RR) if and only if, for all S_1 and S_2 in K such that $S_1 \subset S_2$ and $S_1 \cap C(S_2) \neq \phi$, $S_1 \cap C(S_2) = C(S_1)$ holds true. (In other words, it is required that if some elements are chosen out of S_2, and then the range of alternatives is narrowed to S_1 but still contains some previously chosen elements, no previously rejected element becomes chosen and no previously chosen element becomes unchosen.) This *Arrovian property* (A) is to be decomposed into what we call the *Bordesian property* (B) and the *Chernovian property* (C). We say that C satisfies (B) (resp. (C)) if and only if, for all S_1 and S_2 in K such that $S_1 \subset S_2$ and $S_1 \cap C(S_2) \neq \phi$, $S_1 \cap C(S_2) \supset C(S_1)$ (resp. $S_1 \cap C(S_2) \subset C(S_1)$). The property (B) requires that if some elements are chosen out of S_2 and then the range of alternatives is narrowed to S_1 but still contains some previously chosen elements, no previously unchosen element becomes chosen. The property (C) requires, on the other hand, that if some elements of subset S_1 of S_2 are chosen from S_2, then they should be chosen from S_1. The property (B) is due to Bordes (1976), while the property (C) is named after Chernoff (1954), although in the present context it is better known as Sen's condition α (see Sen, 1969, 1970b).

An alternative formulation of the concept of rational choice goes as follows. Given a preference relation R and an $S \in K$, we define:

$$M_R(S) = \{x : x \in S \text{ and not } yPx \text{ for all } y \in S\}.$$

We say that a choice function C is *M-rational* if there exists a preference relation R, to be called an *M-rationalization*, such that $C(S) = M_R(S)$ for all $S \in K$. In view of some arguments by Herzberger (1973) and Schwartz (1970) in favour of the M-rationality concept, it may be worth our while to investigate how M-rationality will fare in the context of the impossibility results. In order to do so, let C be M-rational with an M-rationalization R. Let us define a binary relation R' by $\{xR'y \leftrightarrow (xRy \text{ or not } yRx)\}$ for all x and y in X. It can easily be shown that $M_R(S) = G_{R'}(S)$ for all $S \in K$. Therefore *if C is M-rational, it is rational.* This being the case, the concept of M-rationality has no special role to play in the impossibility exercises (see, however, Suzumura, 1976a).

So much for rational choice functions. Let us now introduce some axioms of revealed preference. Our *first axiom of revealed preference* (FARP) and *second axiom of revealed preference* (SARP) consider the binary choice of x over y: $\{x\} = C(\{x, y\})$. In this case, (FARP) requires that there should be no choice situation $S \in K$ such that $x \in S$ and $y \in C(S)$, while (SARP) requires that there should be no choice situation $S \in K$ such that $x \in S - C(S)$ and $y \in C(S)$. What these two axioms essentially say is that the choice pattern revealed in binary choice should never be contradicted in non-binary choice. Our third revealed preference axiom, which we called in Blair *et al.* (1976) the *base triple acyclicity* (BTA), is concerned solely with binary choices. It requires that if x is chosen over y in binary choice and y is chosen over z in binary choice, x should never be rejected in the binary choice between x and z: $\{x\}=C(\{x, y\})$ and $\{y\}=C(\{y, z\}) \rightarrow x \in C(\{x, z\})$ for all $x, y, z \in X$. Finally we introduce the *weak axiom of revealed preference* (WARP), due originally to Samuelson (1947), who introduced it in the context of consumers' behaviour. It says that, if $\{x \in C(S)$ and $y \in S-C(S)\}$ for some $S \in K$, then $\{x \in S'$ and $y \in C(S')\}$ for no $S' \in K$. Namely, if in some choice situation x is chosen while y, though available, is rejected, then y should never be chosen in the presence of x.

Essential for our present purpose is an implication network among these various requirements on the choice function, which is summarized in the implication diagram of Figure 1. Here, a double-headed arrow indicates

$$\begin{array}{ccccccc}
 & & (R) & \longrightarrow & & (C) & \\
 & & \uparrow & & & \downarrow & \\
(A) & \leftrightarrow & (RR) & \leftrightarrow (WARP) \rightarrow & & (FARP) & \rightarrow (BTA) \\
 & & \downarrow & & & \downarrow & \\
 & & (B) & \longrightarrow & & (SARP) &
\end{array}$$

FIGURE 1

equivalence, and a single-headed arrow indicates implication. Some of these arrows have been established by Arrow (1959) and Sen (1969), while the remaining ones will be proved in Section V.

II. ARROW'S THEOREM AND COLLECTIVE RATIONALITY

We are now ready to discuss the problem originally posed by Arrow (1963). Suppose that there exist n individuals in the society and let $N=\{1, \ldots, i, \ldots, n\}$ stand for the index set of the individuals. In order to exclude a trivial case, we assume that $n \geqslant 2$. An n-tuple of preference orderings $(R_1, \ldots, R_i, \ldots, R_n)$, one ordering for each individual, will be called a *profile* (of individual preference orderings). Corresponding to R_i, we define P_i by $xP_iy \leftrightarrow (xR_iy$ and not $yR_ix)$. The problem of collective choice is to find a function F, to be called a *collective choice rule* (CCR), which aggregates a profile into a collective choice function:

(3) $\quad C=F(R_1, \ldots, R_i, \ldots, R_n)$.

If an $S \in K$ is specified as an available alternatives set, $C(S)$ represents the set of socially chosen elements from S when the profile $(R_1, \ldots, R_i, \ldots, R_n)$ prevails. In what follows, we will always assume *universal domain* (U) for F: F should be able to aggregate all logically possible profiles. As a matter of terminology, we say that F satisfies the property ω if F always yields a choice function having the property ω. (For example, if C determined by (3) is always rational, we say that F is rational in itself.) Some other conditions on CCR will be introduced when and where necessity dictates.

Arrow (1963) has shown that *there exists no* CCR *that satisfies the universal domain* (U), *the regular-rationality* (RR), *the independence of irrelevant alternatives* (I), *the Pareto rule* (P) *and the non-dictatorship* (D). Here the conditions (I), (P) and (D) are defined as follows. Let x and y be any two alternatives. A

CCR F is said to satisfy the condition (I) if, for any two profiles $(R_1, \ldots, R_n)$ and $(R'_1, \ldots, R'_n)$ such that $(xR_iy \leftrightarrow xR'_iy$ and $yR_ix \leftrightarrow yR'_ix)$ for all $i \in N$, we have $C(\{x, y\}) = C'(\{x, y\})$; the condition (P) if, for any profile $(R_1, \ldots, R_n)$ such that xP_iy for all $i \in N$, we have $\{x\} = C(\{x, y\})$; the condition (D) if there exists no individual $i \in N$ such that, for any profile $(R_1, \ldots, R_n)$, $xP_iy \rightarrow \{x\} = C(\{x, y\})$. Throughout we let $C = F(R_1, \ldots, R_n)$ and $C' = F(R'_1, \ldots, R'_n)$.

Among these Arrow's incompatible conditions on CCR, (P) and (D) can hardly be objectionable, so that the culprit for Arrow's phantom should be sought among (U), (RR) and (I). In what follows, our attention will be focused upon the condition (RR). (The relevance of the condition (U) is extensively discussed in Sen, 1970b, Ch. 10, in the context of simple majority decision, while condition (I) is critically examined by Hansson, 1973.) Can we circumvent Arrow's difficulty by weakening (RR) to some reasonable extent?

In order to prepare for our answer to this question, it is necessary to introduce some more conditions on CCR. First, there is the condition of the *non-weak-dictatorship* (WD). Let x and y be any two alternatives. We say that F satisfies the condition (WD) if there exists no $i \in N$ such that, for any profile $(R_1, \ldots, R_n)$, $xP_iy \rightarrow x \in C(\{x, y\})$. Clearly (WD) is a stronger version of (D). Second, there is the condition of the *positive responsiveness* (PR). Let $i \in N$ be any prescribed individual and let x and y be any two alternatives. We say that F satisfies the condition (PR) if (a) $(R_1, \ldots, R_n)$ is a profile resulting in $x \in C(\{x, y\})$, and (b) $(R'_1, \ldots, R'_n)$ is another profile with $R_j = R'_j$ for all $j \in N - \{i\}$ and $\{(yP_ix$ and $xI'_iy)$ or $(xI_iy$ and $xP'_iy)\}$, then $\{x\} = C'(\{x, y\})$.

We now put forward two theorems which are relevant in the context of the question raised in this section.

Theorem 1. If $n \geqslant 4$, there exists no CCR which satisfies (U), (FARP), (P), (I), (WD) and (PR).

Theorem 2. There exists a CCR which satisfies (U), (SARP), (P), (I), (WD) and (PR).

Note that the impossibility in Theorem 1 is turned into the possibility in Theorem 2 by simply replacing the condition (FARP) by (SARP). Put differently, in the presence of (U), (P), (I), (WD) and (PR), the gulf that separates possibility from impossibility is located by Theorem 1 and Theorem 2 as being in between (FARP) and (SARP).

III. Paradise Lost

It is now time to make some observations by comparing Arrow's theorem, Theorem 1 and Theorem 2. First, let us compare Theorem 1 with Arrow's theorem. In Theorem 1, Arrow's rationality condition (RR) is substantially weakened into (FARP), but (D) is strengthened into (WD), and (PR) (which does not appear in Arrow's theorem) is invoked. Therefore, strictly speaking, Theorem 1 is not a generalization of Arrow's theorem. It may, however, be claimed that (WD) and (PR) are still reasonable conditions on a democratic collective choice rule and our Theorem 1 may be taken to mean that Arrow's difficulty cannot be got rid of even if his rationality condition (RR) is substantially weakened.

Let us next compare Theorem 1 with Theorem 2. We have noticed already that, although (FARP) and (SARP) look quite similar, their implications in the context of collective choice are very disparate. The contrast being sharp, we might be tempted to say that the Arrovian impossibility depended squarely on the unjustifiably strong collective rationality requirement such as (FARP) and that if we replaced (FARP) by a weaker (SARP), the Arrovian phantom would go. Life would be happier then for democrats if this could really be the end of the story. Unfortunately, however, this is not the case. The gist is that the theory of collective choice is full of disturbing paradoxes and Arrow's theorem is only one eminent example.

Two more conditions on the CCR are to be introduced. The first one is a strengthened version of the Pareto rule (P). Let S be any set in K and let $(R_1, \ldots, R_n)$ be any profile. Let y be any point in S. We say that F satisfies the *strong Pareto rule* (SP) if $\{(xP_iy$ for all $i \in N)$ for some $x \in S\} \rightarrow y \notin C(S)$. In words, y should not be chosen out of S if there exists an x in S that is unanimously preferred to y. The second condition is what Sen (1970) called the condition of *minimal liberalism* (ML), which reads as follows. There are at least two individuals such that for each of them there is at least one pair of alternatives over which he is decisive; that is, there is a pair of x, y, such that if he prefers x (respectively y) to y (respectively x), then society should prefer x (respectively y) to y (respectively x) (Sen, 1970a, p. 154).

Sen (1970a) has shown that *there exists no* CCR *that satisfies universal domain* (U), *rationality* (R), *the Pareto rule* (P) *and minimal liberalism* (ML). This so-called *liberal paradox* has been generalized by Batra and Pattanaik (1972), from which it follows that *there exists no* CCR *that satisfies* (U), (SARP), (SP) *and* (ML). This being the case, we cannot but say that, although the replacement of (FARP) by (SARP) fares quite well in exorcising the Arrovian phantom, it cannot let the CCR be free from Sen's liberal paradox. The cloud is thicker here because the independence condition (I) (which has also been suspected to be a possible culprit for the Arrow's difficulty) does not play any role at all in establishing Sen's paradox. Our conclusion is that, even if the collective rationality condition is substantially weakened, we cannot eradicate the paradoxes of democratic decision.

IV. Bergson–Samuelson Social Welfare Ordering

We now turn to discuss the logical foundation of the Bergson–Samuelson theory of Paretian welfare economics. It has long been lamented that Arrow gave his collective choice rule, which is, in our terminology, a regular-rational CCR, the name of *social welfare function*. Clearly it is completely different from the Bergson–Samuelson social welfare function which, according to Little, is "a 'process or rule' which would indicate the best economic state as a function of a changing environment (i.e. changing sets of possibilities defined by different economic transformation functions), *the individual tastes being given*" (Little, 1952, p. 423, Little's italics). It is also claimed that "the only axiom restricting Bergson social welfare function (of individualistic type) is a 'tree' property of Pareto-optimality type" (Samuelson, 1967, p. 49). The purpose of the rest of this paper is to examine the possibility of this *fixed* profile theory of Paretian welfare economics.

Let us, therefore, fix a profile $(R_1, \ldots, R_i, \ldots, R_n)$ of the individual preference relations. Let a binary relation Q, to be called the *Pareto unanimity relation*, be defined by $(xQy \leftrightarrow xR_iy$ for all $i \in N)$ for all $x, y \in X$. The asymmetric component P_Q and the symmetric component I_Q of Q are defined, respectively, by $(xP_Qy \leftrightarrow xQy$ and not $yQx)$ and $(xI_Qy \leftrightarrow xQy$ and $yQx)$.

A *social welfare ordering* (SWO) in the sense of Bergson and Samuelson is an ordering R such that $\{(xQy \rightarrow xRy)$ and $(xP_Qy \rightarrow (xRy$ and not $yRx))\}$ for all $x, y \in X$. (In words, an SWO is an ordering that preserves whatever information the Pareto unanimity relation can tell us about the wishes of the individuals.) A *social welfare function* (SWF) in the sense of Bergson and Samuelson is a numerical representation u of R: $u(x) \geqslant u(y) \leftrightarrow xRy$.

Our problem is to examine the existence of an SWO for a fixed profile. Thanks to the work of Debreu (1959) and others, we know that an SWO may not have an SWF representing it. It may, however, be said that what is important is an SWO but not its numerical representation.

Suppose that $R_1, \ldots, R_n$ are orderings. In this case Q is a quasi-ordering, so that a corollary of Szpilrajn's theorem (Fishburn, 1973, Lemma 15.4) assures us of the existence of an SWO corresponding to the given profile. How about the case where R_i $(i \in N)$ is not necessarily transitive? Generally speaking, we cannot have an SWO in this case, as the following examples where $n = 3$ and $X = \{x, y, z\}$ exhibit.

$$\text{(A)}\ \begin{matrix} xI_1y,\ yI_1z,\ zP_1x \\ yI_2z,\ zI_2x,\ xP_2y \\ zI_3x,\ xI_3y,\ yP_3z \end{matrix} \qquad \text{(B)}\ \begin{matrix} xP_1y,\ yI_1z,\ zI_1x \\ yI_2z,\ zI_2x,\ xP_2y \\ zI_3x,\ xI_3y,\ yP_3z \end{matrix}$$

(Notice here that in both profiles individual strict preference is transitive but individual indifference is not.) In the case (A), we have xP_Qy, yP_Qz and zP_Qx, so that Q is cyclic and there exists no ordering which can subsume this Q. In the case (B), we have xP_Qy, yP_Qz and xI_Qz. Although this Q is acyclic, we cannot still have an ordering subsuming it. Under what condition can Q have an ordering which subsumes it?

Our answer will be given via a general theorem on the extension of a binary relation. Let R be a given binary relation. A t-tuple of alternatives $(x^1, x^2, \ldots, x^t)$ is called a *PR-cycle* of order t if we have $x^1Px^2R \ldots. Rx^tRx^1$, where P is the asymmetric component of R. We say that R is *consistent* if there exists no PR-cycle of any finite order. It is clear that a consistent binary relation is acyclic but not vice versa. An ordering R^* is said to be an *extended ordering* of R if $\{(xRy \rightarrow xR^*y)$ and $(xPy \rightarrow xP^*y)\}$ for all $x, y \in X$, where P^* is the asymmetric component of R^*. We can now state a theorem on the existence of an extended ordering.

Theorem 3. A binary relation R has an extended ordering R^* if and only if R is consistent.

In passing we note that Champernowne (1969) introduced a concept of consistent preference (or probability) relations which is similar to but distinct from ours. We say that a t-tuple $(x^1, x^2, \ldots, x^t)$ is a *C-cycle* of order t if $x^1(P \cup N)x^2R \ldots Rx^tRx^1$, where $\{x^1(P \cup N)x^2 \leftrightarrow (x^1Px^2$ or $x^1Nx^2)\}$, $\{x^1Nx^2 \leftrightarrow$ (not x^1Rx^2 and not $x^2Rx^1)\}$. R is said to be *Champernowne-consistent* if there exists no C-cycle of any finite order. Unfortunately it turns out that R is Champernowne-consistent if and only if it is transitive. Clearly we have only to show that Champernowne consistency implies transitivity. Suppose that R is not transitive. Then we have xRy, yRz but not xRz for some $x, y, z \in X$. Therefore we have $z(P \cup N)xRyRz$, so that (z, x, y) is a C-cycle of order 3. In order to show that our concept of consistency does not reduce to transitivity, we give an example. Let $X=\{x, y, z\}$ and let R be defined by xPy, yRz and xNz.

It follows from Theorem 3 that a social welfare ordering exists for a profile $(R_1, \ldots, R_n)$ if and only if the Pareto unanimity relation Q corresponding to this profile is consistent. In this context, it is worth our while to note that the transitivity of an R_i implies that of the strict preference P_i and of the indifference I_i, and cases against transitive indifference are plenty. This being the case, it is interesting to see that we can define an SWO corresponding to a profile $(R_1, \ldots, R_n)$ even if R_i is not necessarily transitive so far as Q satisfies the axiom of consistency. The contrast between the *variable* profile framework of the Arrow–Sen theory, which leads us to the logical impossibility even under weakened rationality requirement, and the fixed profile framework of the Bergson–Samuelson theory, which can even accommodate some individual preference intransitivity, is made sharper than ever.

V. Proofs

Proof of the implication diagram

First, we show that (C) implies (FARP). Let x and y be such that $C(\{x, y\})=\{x\}$ and take a superset $S \in K$ of $\{x, y\}$. By virtue of (C), we have $\{x, y\} \cap C(S) \subset C(\{x, y\})$, so that $y \notin C(S)$.

Second, we show that (FARP) implies (BTA). If (BTA) is not satisfied by C, there exist x, y and z in X such that $C(\{x, y\})=\{x\}$, $C(\{y, z\})=\{y\}$ and $C(\{x, z\})=\{z\}$. Suppose that C satisfies (FARP). Then x does not belong to $C(\{x, y, z\})$, because $\{z\}=C(\{x, z\})$. Similarly neither y nor z belongs to $C(\{x, y, z\})$. Thus $C(\{x, y, z\})=\phi$, a contradiction. Therefore (FARP) implies (BTA).

Third, it will be shown that (B) implies (SARP). If C does not satisfy (SARP), there exist $x, y \in X$ and $S \in K$ such that $C(\{x, y\})=\{x\}$, $x \in S-C(S)$ and $y \in C(S)$. Let $S'=\{x, y\}$. Then $S' \subset S$, $S' \cap C(S) \neq \phi$, but $x \notin S' \cap C(S)$ and $x \in C(S')$, so that (B) does not hold. Therefore (B) implies (SARP).

In order to show that a single-headed arrow in the diagram cannot in general be reversed, we put forward the following examples.

Example 1. $X=\{x, y, z\}$, $K=\{S_1, \ldots, S_7\}$, $S_1=\{x\}$, $S_2=\{y\}$, $S_3=\{z\}$, $S_4=\{x, y\}$, $S_5=\{y, z\}$, $S_6=\{x, z\}$, $S_7=X$, $C(S_t)=S_t$ $(t=1, 2, 3)$, $C(S_4)=S_1$, $C(S_5)=S_2$, $C(S_6)=S_3$ and $C(S_7)=S_7$.

Example 2. The same as Example 1 except for $C(S_6)=S_6$ and $C(S_7)=S_2$.

Example 3. The same as Example 1 except for $C(S_6)=S_6$ and $C(S_7)=S_1$.

Example 4. The same as Example 1 except for $C(S_6)=S_1$.

Example 5. The same as Example 1 except for $C(S_4)=S_4$, $C(S_5)=S_5$, $C(S_6)=S_6$ and $C(S_7)=S_1$.

Example 6. $X=\{w, x, y, z\}$, $K=\{S_1, \ldots, S_{15}\}$, $S_1=\{x\}$, $S_2=\{y\}$, $S_3=\{z\}$, $S_4=\{w\}$, $S_5=\{w, x\}$, $S_6=\{w, y\}$, $S_7=\{w, z\}$, $S_8=\{x, y\}$, $S_9=\{x, z\}$, $S_{10}=\{y, z\}$, $S_{11}=\{w, x, y\}$, $S_{12}=\{w, x, z\}$, $S_{13}=\{w, y, z\}$, $S_{14}=\{x, y, z\}$, $S_{15}=X$, $C(S_t)=S_t$ $(t=1, \ldots, 10)$, $C(S_{11})=S_8$, $C(S_{12})=S_1$, $C(S_{13})=S_2$, $C(S_{14})=S_1$ and $C(S_{15})=S_2$.

The choice function in Example 1 satisfies (SARP) but not (FARP), so that (SARP) does not necessarily imply (FARP). The choice function in Example 2 satisfies (BTA) but not (FARP), so that (BTA) does not imply (FARP) in general. (Incidentally, Example 1 gives a choice function which satisfies (SARP) but not (BTA), while the choice function in Example 2 satisfies (BTA) but not (SARP). It follows, therefore, that (SARP) and (BTA) are generally independent.) The choice function in Example 3 satisfies (R) without satisfying (RR), so that (R) does not imply (RR). This choice function does not satisfy (B), so that (R) does not imply (B). The choice function in Example 4 satisfies (B) but not (R), so that *a fortiori* it does not satisfy (RR). Thus (RR) is not implied by (B). The choice function given by Example 5 satisfies (C) without satisfying (R), so that (C) does not necessarily imply (R). It can be seen that the choice function in Example 6 satisfies (FARP) but not (C), neither does it satisfy (WARP). Therefore, (FARP) does not imply (C) in general, neither does (FARP) imply (WARP). Finally we note that the choice function in Example 6 satisfies (SARP) but not (B), so that (SARP) does not imply (B) in general. Q. E. D.

Proof of Theorem 1. It has been shown by Blair *et al.* (1976) that, if $n \geqslant 4$, there exists no CCR which satisfies (U), (BTA), (P), (I), (WD) and (PR) (see also Sen, 1975). Our implication diagram shows that (FARP) is a stronger requirement on CCR than (BTA), so that *a fortiori* it is incompatible with (U), (P), (I), (WD) and (PR), establishing Theorem 1. Q. E. D.

Let R and S be, respectively, a binary relation on X and a subset of X. A sequence of relations $\{R_S{}^{(n)}\}_{n=1}^{\infty}$ is then defined recursively by $R_S{}^{(1)} = R$, $R_S{}^{(n)} = RR_S{}^{(n-1)} = \{(x, y) : (x, z) \in R \text{ and } (z, y) \in R_S{}^{(n-1)} \text{ for some } z \in S\}$ $(n \geqslant 2)$. The *transitive closure of R relative to S* is a binary relation which is defined by $T(R|S) = \cup_{n=1}^{\infty} R_S{}^{(n)}$. For simplicity we let $T(R) = T(R|X)$.

Proof of Theorem 2. For any profile $(R_1, \ldots, R_n)$, let $N(xR_iy)$ be the number of individuals who regard x to be at least as good as y. We define an R by $[xRy \leftrightarrow N(xR_iy) \geqslant N(yR_ix)]$ for all $x, y \in X$ and define a CCR F by associating a choice function $C(S) = G_{T(R|S)}(S)$ with $(R_1, \ldots, R_n)$. It is easy to verify that this CCR satisfies (U), (B), (P), (I), (WD) and (PR) (see Bordes, 1976). Our implication diagram shows that (SARP) is a weaker requirement on CCR than (B), so that *a fortiori* it is compatible with the requirements (U), (P), (I), (WD) and (PR). Hence Theorem 2. Q. E. D.

Proof of Theorem 3. (a) *Necessity proof.* Suppose that R has an extended ordering R^*. Let t be any finite positive integer and suppose that we have $x^1Px^2R \ldots Rx^t$ for some $x^1, x^2, \ldots, x^t$ in X. Then we have $x^1P^*x^2R^* \ldots R^*x^t$, which yields $x^1P^*x^t$, thanks to the transitivity of R^*. Thus we have (not $x^tR^*x^1$), which implies (not x^tRx^1). It follows that if R has an extended ordering, it has to be consistent. (b) *Sufficiency proof.* Let the identity Δ be defined by $\Delta = \{(x, x) : x \in X\}$. We define a binary relation Q by

$$Q = \Delta \cup T(R). \tag{4}$$

We show that Q is a quasi-ordering. Reflexivity is obvious. In order to show its transitivity, let $(x, y), (y, z) \in Q$. If $(x, y), (y, z) \in T(R)$, we have $(x, z) \in T(R) \subset Q$. If $(x, y) \in \Delta$ (resp. $(y, z) \in \Delta$), we have $z = y$ (resp. $y = z$), so that $(x, z) \in Q$ follows from $(y, z) \in Q$ (resp. $(x, y) \in Q$). Q being a quasi-ordering, it has an extension that is an ordering (see, for example, Fishburn, 1973, Lemma 15.4). If we can show that Q is an extension of R, we are home. For that purpose, we have to show that R is included in Q and P in P_Q (asymmetric component of Q). The former is obvious. To prove the latter, assume $(x, y) \in P$, which means $(x, y) \in R$ and $(y, x) \notin R$. From $(x, y) \in R$ it follows that $(x, y) \in Q$, so that we have only to prove that $(y, x) \notin Q$. Assume, therefore, that $(y, x) \in Q$. Clearly $(y, x) \notin \Delta$, otherwise we cannot have $(x, y) \in P$. It follows that $(y, x) \in T(R)$. When $(x, y) \in P$ is added to this, we obtain a PR-cycle, and this contradiction proves the theorem. Q. E. D.

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